Discrete Time State Space Analysis of Electrical Networks

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Abstract - **Linear system techniques can be applied for the analysis of electromagnetic transient phenomena, harmonics or the calculation of reduced order equivalents. These techniques are usually based on a continuous time state space description of the electrical network. However, the definition of state variables in a meshed network is a difficult task because the physical storage quantities, such as inductive currents and capacitive voltages, generally don't lead to an independent set of state variables. This paper describes a discrete time state space approach allowing to calculate linear system characteristics without the necessity of identifying independent state variables.**

Keywords: difference equations, network equivalents, harmonics analysis, real time simulation

I. Introduction

Linear system techniques have successfully been applied for the analysis of electromagnetic transient phenomena, particularly in the context of reduced order equivalent systems, harmonics or very fast simulation algorithms for real time applications. These techniques are based on a state space description of the electrical network. However, the definition of state variables in a meshed electrical network is a difficult task because the physical storage quantities, such as inductive currents and capacitive voltages, generally don't lead to an independent set of state variables.

Different approaches to overcome this problem have been described at previous IPST conferences, e.g. the *Modified Nodal Voltage Approach*[1] or the *Augmented State Space Formulation*[2]. Both approaches are based on a continuous time description of the electrical network and require therefore approximations for the consideration of distributed parameter line models. This paper proposes an alternative approach based on a state space analysis in the discrete time domain.

Discrete time difference equations are widely used in simulation programs for analyzing power systems transients. The most popular approach is the difference conductance method, first implemented in the EMTP by Dommel [3] in the early seventies. The greatest advantage of difference equation methods is the simplicity of the solution process: By converting the differential equation of each branch element to a discrete time equation using e.g. the trapezoidal rule, each branch element can be described by either an equivalent current source with a parallel conductance or by an equivalent voltage source with series resistance.

In this paper, it is first described, how the difference equations of the various branch elements can be combined to a discrete time state space system for the entire network. Then, a modal transformation is applied to the discrete time system by calculating the corresponding eigenvalues and eigenvectors. It is then analyzed, how eigenvalues and eigenvectors are distorted by the discretisation with finite step size. Using the obtained relationship between ideal and distorted eigenvalues, the exact eigenvalues of the original, continuous time system can be regained. The inclusion of distributed parameter line models into the discrete time state space approach completes the description of the methodology. The paper concludes by discussing the benefits and disadvantages of the presented approach compared to classical, continuous time methods, specially with regard to precision and computational efficiency when distributed parameter line models need to be considered.

II. Discretization of Branch Elements

Linear, first order branch elements of electrical networks can generally be described by implicit differential equations of the form:

$$y(t) = F\dot{x}(t) + Hx(t) \tag{1}$$

In case of inductive branch elements, equation (1) has the form of a voltage equation:

$$v(t) = L\frac{di}{dt} + Ri(t) + v_s(t) \tag{2}$$

Capacitive branch elements are directly resulting in current equations:

$$i(t) = C\frac{dv}{dt} + Gv(t) + i_s(t) \tag{3}$$

For describing an electrical network as a set of discrete time difference equations, these branch equations have first to be transformed into difference equations and then combined to an equation system for the entire network.

Basically, any implicit numerical integration method of the form

$$\dot{x}(t+h) = \Omega x(t+h) - \Omega h_x(t+h) \tag{4}$$

$$h_x(t+h) = f\left(x(t), \ldots, x(t-nh), \dot{x}(t), \ldots, \dot{x}(t-mh)\right) \tag{5}$$

can be used for discretizing the continuous time branch equations (2) or (3) (e.g. [4]).

The expression $h_x(t)$ is the so called *history* term and comprises values of x and $\dot{x}$ at previous time steps. Ω is a constant that is usually inverse proportional to the discretisation step size h and has therefore the unit of a frequency.

In case of the *Implicit Euler* method, Ω and $h_x(t)$ are defined as follows:

$$\Omega = \frac{1}{h} \tag{6}$$

$$h_x(t+h) = x(t) \tag{7}$$

The *Trapezoidal Rule* is defined by setting:

$$\Omega = \frac{2}{h} \tag{8}$$

$$h_x(t+h) = x(t) + \Omega^{-1}\dot{x}(t) \tag{9}$$

The first order branch equations (2) and (3) can now be descretized by replacing the derivatives with the approximative formula (4):

$$v(t+h) = L\Omega\left(i(t+h) - h_i(t+h)\right) + Ri(t+h) + v_s(t+h) \tag{10}$$

$$i(t+h) = C\Omega\left(v(t+h) - h_v(t+h)\right) + Gv(t+h) + i_s(t+h) \tag{11}$$

Or, by reordering (10) and (11):

$$v = \underbrace{(L\Omega + R)}_{Z_L} i - \underbrace{L\Omega}_{X_L} h_i + v_s \tag{12}$$

$$i = \underbrace{(C\Omega + G)}_{Y_C} v - \underbrace{C\Omega}_{B_C} h_v + i_s \tag{13}$$

The time arguments can be omitted, because (10) and (11) are valid for every moment in time. Any time dependence and consequently any storage property occurs only in the calculation of the history terms. Hence, the equations (12) and (13) can be interpreted as equivalent voltage and equivalent current sources respectively (see figure 1).
It is also important to mention that the representation according to (12), (13) is independent of the applied discretization method.

III. The Modified Nodal Approach

There are different possibilities for combining the equivalent voltage and current source equations to an equation system for the entire network. The most popular approach is certainly the *Nodal Voltage Approach*. Here, the vector of unknowns consists of all nodal voltages and the resulting system matrix is equivalent to the complex node admittance matrix that is well known from steady state network analysis.

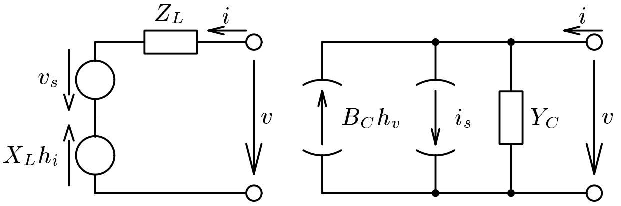

Fig. 1. Equivalent voltage- and current source representation

However, with respect to the discrete time state space representation, it is advantageous to build an equation system, in which all storage variables, such as inductive currents and capacitive voltages appear in the vector of unknowns.

Using the *Modified Nodal Approach* [5], a vector of unknowns can be defined that consists of all node voltages and all inductive currents. This vector of unknowns does not build a *minimum* set of state variables, but it comprises all storage quantities of the system.

The *Modified Nodal Approach* (MNA) can be derived by writing Kirchoffs current law in the following form:

$$\boldsymbol{K}\boldsymbol{i} = \boldsymbol{0} \tag{14}$$

The matrix $\boldsymbol{K}$ is the node-branch incidence matrix of the network and $\boldsymbol{i}$ is the vector of all branch currents (inductive and capacitive).
The relationship between the vector of branch voltages $\boldsymbol{v}$ and the vector of node voltages $\boldsymbol{v}_n$ is the following:

$$\boldsymbol{v} = \boldsymbol{K}^T \boldsymbol{v}_n \tag{15}$$

The vector of branch currents $\boldsymbol{i}$, as well as the node-branch incidence matrix is now partitioned into a submatrix describing all capacitive elements and a submatrix describing all inductive elements:

$$\begin{bmatrix} \boldsymbol{K}_C & \boldsymbol{K}_L \end{bmatrix} \begin{bmatrix} \boldsymbol{i}_C \\ \boldsymbol{i}_L \end{bmatrix} = \boldsymbol{0} \tag{16}$$

Replacing $\boldsymbol{i}_C$ in (16) by (13) and applying the relationship between branch and node voltages (15) leads together with the equations for inductive branch elements (12) to the following two matrix-equations:

$$\underbrace{\boldsymbol{K}_C \boldsymbol{Y}_C \boldsymbol{K}_C^T}_{\boldsymbol{Y}_{nC}} \boldsymbol{v}_n - \boldsymbol{K}_C \boldsymbol{B}_C \boldsymbol{h}_v + \boldsymbol{K}_C \boldsymbol{i}_s + \boldsymbol{K}_L \boldsymbol{i}_L = \boldsymbol{0} \tag{17}$$

$$-\boldsymbol{K}_L^T \boldsymbol{v}_n + \boldsymbol{Z}_L \boldsymbol{i}_L - \boldsymbol{X}_L \boldsymbol{h}_{iL} + \boldsymbol{v}_s = \boldsymbol{0} \tag{18}$$

Applying (15) also to the voltage history terms:

$$\boldsymbol{h}_v = \boldsymbol{K}_C^T \boldsymbol{h}_{vn}$$

the *MNA* system can be written in the following matrix-vector form:

$$\underbrace{\begin{bmatrix} \boldsymbol{Y}_{nC} & \boldsymbol{K}_L \\ -\boldsymbol{K}_L^T & \boldsymbol{Z}_L \end{bmatrix}}_{\boldsymbol{H}_n} \begin{bmatrix} \boldsymbol{v}_n \\ \boldsymbol{i}_L \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{K}_C \boldsymbol{B}_C \boldsymbol{K}_C^T & 0 \\ 0 & \boldsymbol{X}_L \end{bmatrix}}_{\boldsymbol{H}_h} \begin{bmatrix} \boldsymbol{h}_{v_n} \\ \boldsymbol{h}_{iL} \end{bmatrix} - \begin{bmatrix} \boldsymbol{K}_C \boldsymbol{i}_s \\ \boldsymbol{v}_s \end{bmatrix} \tag{19}$$

The system matrix $\boldsymbol{H}_n$ of the *MNA* is a hybrid matrix of real numbers consisting of difference conductances $\boldsymbol{Y}_C$, difference resistances $\boldsymbol{Z}_L$ and some elements of the node-branch incidence matrix.

The hybrid matrix $\boldsymbol{H}_h$ relates the history terms $\boldsymbol{h}_{iL}$ and $\boldsymbol{h}_{vn}$ to the equation system.

IV. EXPLICIT DIFFERENCE EQUATION SYSTEM

For the calculation of eigenvalues and eigenvectors, the implicit difference equation system according to (19) must be transformed to an explicit system of equations:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}_d x(n) + \boldsymbol{B}_d \boldsymbol{w}(n+1) \tag{20}$$

In this discrete time state space system, the state vector is built by $\boldsymbol{x}$. The vector $\boldsymbol{w}$ is a vector of input variables, which are are supposed to be known at each time step.

The transformation of (19) to the explicit form according to (20) consists basically of replacing the history terms by the values of the state vector at previous time steps. Hence, the explicit form depends on the actual discretization method.

A. State Space Form with the Implicit Euler Method

When using the *Implicit Euler Method* for discretizing the branch equations, the explicit form can be obtained without any difficulties by replacing the history terms in (19) with (7). The resulting implicit difference equation system is:

$$\boldsymbol{H}_n \begin{bmatrix} \boldsymbol{v}_n(t+h) \\ \boldsymbol{i}_L(t+h) \end{bmatrix} = \boldsymbol{H}_h \begin{bmatrix} \boldsymbol{v}_n(t) \\ \boldsymbol{i}_L(t) \end{bmatrix} - \begin{bmatrix} \boldsymbol{K}_C \boldsymbol{i}_s(t+h) \\ \boldsymbol{v}_s(t+h) \end{bmatrix} \tag{21}$$

Inverting $\boldsymbol{H}_n$ leads to an explicit difference equation system with the following definitions according to (20):

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{v}_n \\ \boldsymbol{i}_L \end{bmatrix} \tag{22}$$

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{i}_s \\ \boldsymbol{v}_s \end{bmatrix} \tag{23}$$

$$\boldsymbol{A}_d = \boldsymbol{H}_n^{-1} \boldsymbol{H}_h \tag{24}$$

$$\boldsymbol{B}_d = -\boldsymbol{H}_n^{-1} \begin{bmatrix} \boldsymbol{K}_C & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{25}$$

B. State Space Form with the Trapezoidal Rule

When the *Trapezoidal Rule* is used for building the explicit difference equation system, the procedure is slightly more complicated than in case of the *Implicit Euler Method*, because the derivatives $\dot{\boldsymbol{x}}(t)$ appear in the formula for calculating the history terms (9).

In [4], a method is proposed in which the derivatives $\dot{\boldsymbol{x}}(t)$ are replaced by the first order branch equations (2) and (3). However, this approach requires *explicit* branch equations, which can lead to difficulties in case of more complicated power systems elements.

Alternatively, the history terms $\boldsymbol{h}_{iL}$ and $\boldsymbol{h}_{vn}$ can be used as discrete time state variables instead of the physical variables $\boldsymbol{v}_n$ and $\boldsymbol{i}_L$. The great advantage of this approach is that the first order branch equations can remain in their implicit form (2) and (3). At any time step, the physical quantities can easily be regained from their corresponding history values.

For deriving the explicit difference equation system based on the *Trapezoidal Rule*, equation (4), which is valid at any time step is rewritten for time step t. Together with the formula for calculating the history term $h_x(t+h)$, the trapezoidal rule can be expressed as follows:

$$\dot{x}(t) = \Omega x(t) - \Omega h_x(t) \tag{26}$$

$$h_x(t+h) = x(t) + \Omega^{-1} \dot{x}(t) \tag{27}$$

From these two equations, the vector of derivatives $\dot{\boldsymbol{x}}(t)$ can be eliminated resulting in:

$$h_x(t+h) = 2x(t) - h_x(t) \tag{28}$$

The *MNA* system according to (19) can now be transformed into a discrete time state space system by solving (19) for $\boldsymbol{v}_n$ and $\boldsymbol{i}_l$ and inserting the resulting expression into (28):

$$\begin{bmatrix} \boldsymbol{v}_n \\ \boldsymbol{i}_L \end{bmatrix} = \boldsymbol{H}_n^{-1} \boldsymbol{H}_h \begin{bmatrix} \boldsymbol{h}_{vn} \\ \boldsymbol{h}_{iL} \end{bmatrix} - \boldsymbol{H}_n^{-1} \begin{bmatrix} \boldsymbol{K}_C \boldsymbol{i}_s \\ \boldsymbol{v}_s \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} \boldsymbol{h}_{vn}(t+h) \\ \boldsymbol{h}_{iL}(t+h) \end{bmatrix} = \left(2\boldsymbol{H}_n^{-1} \boldsymbol{H}_h - \boldsymbol{I}\right) \begin{bmatrix} \boldsymbol{h}_{vn}(t) \\ \boldsymbol{h}_{iL}(t) \end{bmatrix} - 2\boldsymbol{H}_n^{-1} \begin{bmatrix} \boldsymbol{K}_C \boldsymbol{i}_s(t) \\ \boldsymbol{v}_s(t) \end{bmatrix} \tag{30}$$

This equation system is equivalent to a discrete time state space system according to:

$$\boldsymbol{x}(n+1) = \boldsymbol{A}_d \boldsymbol{x}(n) + \boldsymbol{B}_d \boldsymbol{w}(n) \tag{31}$$

The difference between the forms (31) and (20) is in the consideration of the input vector $\boldsymbol{w}$ at different time steps, which doesn't cause any major difficulties but which must be considered in the analytical solution.

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{h}_{vn} \\ \boldsymbol{h}_{iL} \end{bmatrix} \tag{32}$$

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{i}_s \\ \boldsymbol{v}_s \end{bmatrix} \tag{33}$$

$$\boldsymbol{A}_d = 2\boldsymbol{H}_n^{-1} \boldsymbol{H}_h - \boldsymbol{I} \tag{34}$$

$$\boldsymbol{B}_d = -2\boldsymbol{H}_n^{-1} \begin{bmatrix} \boldsymbol{K}_C & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{35}$$

The vector of node voltages $\boldsymbol{v}_n$ and inductive branch currents $\boldsymbol{i}_L$ can be calculated by using (28):

$$\begin{bmatrix}\boldsymbol{v}_n(t)\\ \boldsymbol{i}_L(t)\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}\boldsymbol{h}_{vn}(t+h)\\ \boldsymbol{h}_{iL}(t+h)\end{bmatrix} + \begin{bmatrix}\boldsymbol{h}_{vn}(t)\\ \boldsymbol{h}_{iL}(t)\end{bmatrix}\right) \tag{36}$$

V. Approximate Solution of the Homogeneous System

The solution of the discrete time state space equation system (20) is (e.g. [6]): [1]

$$\boldsymbol{x}(k) = \boldsymbol{A}_d^k \boldsymbol{x}(0) + \sum_{j=0}^{k-1} \boldsymbol{A}_d^{k-j-1} \boldsymbol{B}_d \boldsymbol{w}(j) \tag{37}$$

The transition matrix $\boldsymbol{A}_d$ can either be calculated directly by multiplying the system matrix $\boldsymbol{A}_d$ k times, or by calculating $\boldsymbol{A}_d^k$ in the modal domain:

$$\boldsymbol{A}_d^k = \boldsymbol{V}\begin{bmatrix} z_1^k & 0 & \dots & 0 \\ 0 & z_2^k & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & z_n^k \end{bmatrix}\boldsymbol{V}^{-1} \tag{38}$$

The values z_i are the eigenvalues of $\boldsymbol{A}_d$.

For evaluating the precision of the discrete time solution, (38) is compared to the the solution of an equivalent, continuous time system:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t-t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t-\tau)\boldsymbol{B}\boldsymbol{w}(\tau)d\tau \tag{39}$$

In case of linear, time invariant systems of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{w} \tag{40}$$

the transition matrix $\boldsymbol{\Phi}(t-t_0)$ is defined by:

$$\boldsymbol{\Phi}(t-t_0) = \mathrm{e}^{\boldsymbol{A}(t-t_0)} \tag{41}$$

Assuming that a set of linear independent eigenvectors exists and setting $h = t - t_0$, the transition matrix $\boldsymbol{\Phi}(h)$ can be calculated as follows:

$$\boldsymbol{\Phi}(h) = \boldsymbol{V}\begin{bmatrix} \mathrm{e}^{\lambda_1 h} & 0 & \dots & 0 \\ 0 & \mathrm{e}^{\lambda_2 h} & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \mathrm{e}^{\lambda_n h} \end{bmatrix}\boldsymbol{V}^{-1} \tag{42}$$

Comparing (38) and (42), the following well known relation between discrete time and continuous time eigenvalues can be found:

$$z_i \approx \mathrm{e}^{\lambda_i h} \tag{43}$$

It has been assumed that the eigenvectors of the discrete time and the original, continuous time system are equal. This will be verified later in this paper.

VI. Exact Solution of the Homogeneous System

Outgoing from a continuous time state space system according to (40) a procedure for calculating the exact transition matrix $\boldsymbol{\Phi}(h)$ from the discrete time system matrix $\boldsymbol{A}_d$ will be derived in this section.

A. Implicit Euler Method

Applying the *implicit Euler method* to the homogenous part of the continuous time state space system results in:

$$\Omega\left(\boldsymbol{x}(t+h) - \boldsymbol{x}(t)\right) = \boldsymbol{A}\boldsymbol{x}(t+h) \tag{44}$$

Solving (44) for $\boldsymbol{x}(t+h)$, the following relation between $\boldsymbol{A}_d$ and $\boldsymbol{A}$ can be found:

$$\boldsymbol{A}_d = \Omega\left(\Omega\boldsymbol{I} - \boldsymbol{A}\right)^{-1} \tag{45}$$

Each corresponding eigenvalue and eigenvector of $\boldsymbol{A}$ is defined by the following equation:

$$\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i \tag{46}$$

Consequently, λ_i and $\boldsymbol{v}_i$ also comply with the following equation:

$$\left(\Omega\boldsymbol{I} - \boldsymbol{A}\right)\boldsymbol{v}_i = \left(\Omega - \lambda_i\right)\boldsymbol{v}_i \tag{47}$$

Multiplying (47) by $\Omega\left(\Omega\boldsymbol{I} - \boldsymbol{A}\right)^{-1}$ and $1/\left(\Omega - \lambda_i\right)$ leads to the following relation between discrete time and continuous time eigenvalues:

$$\underbrace{\frac{\Omega}{\Omega - \lambda_i}}_{z_i}\boldsymbol{v}_i = \underbrace{\Omega\left(\Omega\boldsymbol{I} - \boldsymbol{A}\right)^{-1}}_{\boldsymbol{A}_d}\boldsymbol{v}_i \tag{48}$$

Hence, the original eigenvalues can be calculated from the discrete time eigenvalues with:

$$\lambda_i = \Omega\frac{z_i - 1}{z_i} \tag{49}$$

According to (48), the eigenvectors of the discrete time system are equal to the original eigenvectors. Therefore, by using (49), the exact transition matrix (42) can be built just by using the eigenvalues z_i of the discrete time system. The main diagonal elements of the transition matrix (42) are then defined by:

$$\mathrm{e}^{\lambda_i h} = \mathrm{e}^{\frac{z_i - 1}{z_i}\Omega h} \tag{50}$$

The approach described in this section is based on the assumption that the vector $\boldsymbol{x}$ consists of linear independent state variables. When starting from a difference equation approach however, the elements of $\boldsymbol{x}$ are usually not linear independent. The corresponding continuous time system is hence a mixed system consisting of differential and algebraic equations.

However, the discrete time system defined by the matrix $\boldsymbol{A}_d$ and consequently the transition matrix $\boldsymbol{\Phi}(h)$ according to (50) can be built without any problems, even without linear independent state variables. As

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[1] The system according to (31) can be solved by setting $\boldsymbol{w}(j-1)$ in (37)

a result, the method for analytically solving differential equation systems that is presented here, does not require any set of linear independent state variables, which highly simplifies the solution process.

But how does the transition matrix looks like in case of algebraic equations? This question can be answered by considering that each dependency between state variables results in a zero eigenvalue $z_i = 0$ of the matrix $\boldsymbol{A}_d$. The corresponding elements in the modal transition matrix $\mathrm{e}^{\lambda_i t}$ are then also equal to zero because of:

$$\lim_{z_i \to 0} \mathrm{e}^{\frac{z_i - 1}{z_i} \Omega h} = 0 \tag{51}$$

The validity of the general approach (37), also in case of mixed algebraic-differential equations can be shown using singular perturbation considerations (e.g. [7]).

B. Trapezoidal Rule

A similar approach leads to the following equation for calculating the continuous time eigenvalues based on a difference equation system obtained with the *trapezoidal rule*:

$$\lambda_i = \Omega \frac{z_i - 1}{z_i + 1} \tag{52}$$

As before, continuous time and discrete time eigenvectors are equal.

VII. Inclusion of Distributed Parameter Line Models

The particularity of distributed line models is the ideal time delay involved. A frequency independent, single phase line can be described with the following equations:

$$v_1(t) = v_{h2}(t - \tau) + Z_c i_1(t) \tag{53}$$

$$v_2(t) = v_{h1}(t - \tau) + Z_c i_2(t) \tag{54}$$

The line history voltages $v_{h1}(t)$ and $v_{h1}(t)$ are defined as follows:

$$v_{h1}(t) = a\big(v_1(t) + Z_c i_1(t)\big) \tag{55}$$

$$v_{h2}(t) = a\big(v_2(t) + Z_c i_2(t)\big) \tag{56}$$

The time constant τ of the ideal time delay is the travel time of the line, Z_c is the surge impedance and a represents a frequency independent damping.

Frequency dependencies of surge impedance Z_c and transmission coefficient a are usually approximated by rational transfer functions (e.g. [8]), which can be discretized and included in the discrete time state space system as shown in the previous sections.

In continuous time approaches, the ideal time delay needs to be approximated, e.g. by the Padé approximation, as suggested in [2]. Alternatively, network line models, such as equivalent Π- or T-sections or pole-fitting approaches as described in [9] can be used.

In a discrete time state space system, however, ideal travel times can ideally be considered, if the travel times of the modeled lines are integer multiples of the discretization step size.

Line equations can then be included in the *MNA* as equivalent voltage sources. But in contrast to inductive branch elements, the history voltages v_{h1} and v_{h2} of the line equations depend on voltage *and* current.

The procedure of including distributed parameter line models into the discrete time state space formulation will be shown with the following example.

VIII. Example

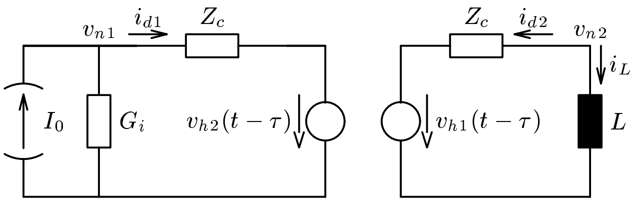

Fig. 2. Example network

The configuration according to figure 2 is used to demonstrate the inclusion of distributed parameter line models into the discrete time state space system.

First, the line history voltages need to be included in the vector of state variables. The vector of unknowns consists then of the nodal voltages $\boldsymbol{v}_n$, the currents of inductive branch elements $\boldsymbol{i}_L$ and the branch currents of distributed parameter line models $\boldsymbol{i}_d$. The *MNA* system of the configuration according to figure 2 can be expressed as follows:

$$\begin{bmatrix} G_i & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & Z_L & 0 & 0 \\ -1 & 0 & 0 & Z_c & 0 \\ 0 & -1 & 0 & 0 & Z_c \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \\ i_L \\ i_{d1} \\ i_{d2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{vn1} \\ h_{vn2} \\ h_{iL} \\ h_{id1} \\ h_{id2} \end{bmatrix} + \begin{bmatrix} I_0 \\ 0 \\ 0 \\ -v_{h2}(t-\tau) \\ -v_{h1}(t-\tau) \end{bmatrix} \tag{57}$$

For transforming (57) into a discrete time state space system, the line-history terms have to be replaced. This can ideally be done if the travel time τ of the line is an integer multiple of the discretisation step size h.

The travel time requires $2k$ additional state variables with $k = \tau/h$. Using in this example $\tau = 2h$, four more state variables have to be added:

$$s_{11}(t + h) = v_{h1}(t) \tag{58}$$

$$s_{12}(t + h) = s_{11}(t) \tag{59}$$

and for the other side of the line:

$$s_{21}(t + h) = v_{h2}(t) \tag{60}$$

$$s_{22}(t + h) = s_{21}(t) \tag{61}$$

The next step consists of eliminating v_{h1} and v_{h2} from (57), (58) and (60), using the line equations (55) and

(56) that can be expressed in the following matrix-vector form (with $a = 1$, ideal line):

$$\begin{bmatrix} v_{h1} \\ v_{h2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & Z_c & 0 \\ 0 & 1 & 0 & 0 & Z_c \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \\ i_L \\ i_{d1} \\ i_{d2} \end{bmatrix} \tag{62}$$

Together with the node incidence matrix of line models $\boldsymbol{K}_d$ and the branch impedance matrix $\boldsymbol{Z}_c$ of all distributed parameter line models (62) can be rewritten as follows:

$$\begin{bmatrix} v_{h1} \\ v_{h2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}_d^T & \boldsymbol{Z}_c \end{bmatrix} \begin{bmatrix} v_{n1} \\ v_{n2} \\ i_L \\ i_{d1} \\ i_{d2} \end{bmatrix} \tag{63}$$

From (63), the vector of unknowns can be eliminated using (57):

$$\begin{bmatrix} v_{h1}(t) \\ v_{h2}(t) \end{bmatrix} = \begin{bmatrix} s_{11}(t+h) \\ s_{21}(t+h) \end{bmatrix} = \left(\begin{bmatrix} \boldsymbol{K}_d^T & \boldsymbol{Z}_c \end{bmatrix} \boldsymbol{H}_n^{-1} \right) \left(\boldsymbol{H}_h \begin{bmatrix} h_{vn1}(t) \\ h_{vn2}(t) \\ h_{iL}(t) \\ h_{id1}(t) \\ h_{id2}(t) \end{bmatrix} + \begin{bmatrix} I_0 \\ 0 \\ 0 \\ -s_{22}(t) \\ -s_{12}(t) \end{bmatrix} \right) \tag{64}$$

Replacing the vector of node voltages and branch currents in (57) by the corresponding history values using (28), (59) and (61) leads finally to the explicit state space system:

$$\begin{bmatrix} h_{vn1}(t+h) \\ h_{vn2}(t+h) \\ h_{iL}(t+h) \\ h_{id1}(t+h) \\ h_{id2}(t+h) \end{bmatrix} = \left(2\boldsymbol{H}_n^{-1} \boldsymbol{H}_h - \boldsymbol{I} \right) \begin{bmatrix} h_{vn1}(t) \\ h_{vn2}(t) \\ h_{iL}(t) \\ h_{id1}(t) \\ h_{id2}(t) \end{bmatrix} - 2\boldsymbol{H}_n^{-1} \begin{bmatrix} -I_0 \\ 0 \\ 0 \\ s_{21}(t) \\ s_{11}(t) \end{bmatrix} \tag{65}$$

The equations (64) and (65), together with (59) and (61) are building an explicit discrete time state space system with state vector $(h_{v1n}, h_{v2n}, h_{iL}, h_{id1}, h_{id2}, s_{11}, s_{12}, s_{21}, s_{22})$.

IX. CONCLUSIONS

In case of linear networks, the explicit difference equation system offers the great advantage of solving the system directly, without the necessity to calculate every single step recursively. Hence, simulations with larger step size can be performed while keeping the discretisation error small leading to extremely fast algorithms as required by real time transient simulators. Additionaly, the paper has shown, how the solution of the original, continuous time system can be found by calculating eigenvalues and eigenvectors of the corresponding discrete time system. Because the eigenvalue mapping is exact, this solution is completely free of any discretization error. An additional advantage of the explicit form is the ability of expressing system characteristics by means of eigenvalues and eigenvectors, which can further be used for system reduction or harmonics analysis.

Finally, distributed parameter line models have been included. Those line models can ideally be represented in a difference equation system if the travel time of the line is an integer multiple of the discretisation step size.

It is evident, that the techniques presented are related to linear or linearized systems. However, many transient problems in electrical networks can be solved with linear power systems elements. A possible way of considering non linear elements could be the approximation of nonlinear characteristics by piece wise linear functions.

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